On a certain rational approximation of arctanh x*

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(Received March 7, 1967)

1. Introduction. In some physical problems, it is necessary to have some rational approximation formulas to the function

$$f(x)=\frac{1}{2}+\frac{1-x^2}{4x}\log\left|\frac{1+x}{1-x}\right|, \qquad 0\leqq x<\infty . \tag{1.1}$$

It is known (e. g. [2]) that the function

$$\text{arctanh } x=\frac{1}{2}\log\frac{1+x}{1-x}, \qquad 0\leqq x\leqq 1 \tag{1.2}$$

has a continued fraction expansion

$$\frac{x|}{|1}-\frac{1^2x^2|}{|3+x^2}-\frac{3^2x^2|}{|5+3x^2}-\frac{5^2x^2|}{|7+5x^2}-\cdots . \tag{1.3}$$

However, the finite approximation of (1.3) reduces to the partial sum of Taylor series of (1.2), viz.

$$x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots+\frac{x^{2n+1}}{2n+1}+\cdots .$$

In the present report, the author would like to give the continued fraction and another numerical rational approximation to the function (1.1).

This work was begun upon a request from Dr. H. Takahasi (Nuclear Engineering Department, Brookhaven National Laboratory), and the author is grateful to the computer center of Brookhaven National Laboratory for some numerical computations.

2. Continued Fraction. In general, a formal power series

$$\sum_{n=1}^{\infty} a_n x^n \tag{2.1}$$

is expanded into continued fraction

* This work was performed under the auspices of the U. S. Atomic Energy Commission, and was published as Brookhaven National Laboratory Technical Report BNL 10938, AMD 447 in January 1967.

$$\frac{\alpha_1 x|}{|1} + \frac{\alpha_2 x|}{|1} + \cdots + \frac{\alpha_n x|}{|1} + \cdots \tag{2.2}$$

by the following "quotient-difference algorithm" (e.g., cf. Henrici [1]). Starting the sequence $\{a_n\}$, we define successively

$$\begin{aligned} &q_n^{(1)} = a_{n+1}/a_n, \\ &e_n^{(k)} = q_{n+1}^{(k)} - q_n^{(k)} + e_{n+1}^{(k-1)}, \\ &q_n^{(k+1)} = q_{n+1}^{(k)} e_{n+1}^{(k)} / e_n^{(k)} \qquad (k, n = 1,2,3, \cdots) . \end{aligned} \tag{2.3}$$

Here we put $e_n^{(0)} = 0$, and assume that no denominator vanishes in order to continue the process indefinitely. Then the coefficients α_n in (2.2) are given by

$$\alpha_1 = a_1, \quad \alpha_{2k} = -q_0^{(k)}, \quad \alpha_{2k+1} = -e_0^{(k)} \qquad (k=1,2,3, \cdots) . \tag{2.4}$$

Now, let us apply the general theorem to the function

$$\varphi(x) = 1 - f(x) = \frac{1}{2}\left(1 - \frac{1-x^2}{2x} \operatorname{arctanh} x\right), \tag{2.5}$$

which has the Maclaurin expansion

$$\varphi(x) = \frac{x^2}{3} + \frac{x^4}{15} + \frac{x^6}{35} + \cdots + \frac{x^{2n}}{4n^2-1} + \cdots . \tag{2.6}$$

Replacing x^2 in (2.6) by x and applying the process (2.3) for

$$a_n = \frac{1}{4n^2-1},$$

it is not difficult to see by induction that

$$\begin{aligned} q_n^{(k)} &= \frac{(2n+2k-3)(2n+2k-1)}{(2n+4k-3)(2n+4k-1)}, \\ e_n^{(k)} &= \frac{2k(2k+2)}{(2n+4k-1)(2n+4k+1)} . \end{aligned} \tag{2.7}$$

Hence the coefficients of the continued fraction associated to (2.6) are given by

$$\alpha_1 = \frac{1}{3}, \quad \alpha_n = -\frac{n^2-1}{4n^2-1} \qquad (n \geqq 2) \tag{2.8}$$

where the latter is valid both for even and for odd numbers n. Multiplying suitable constants, the continued fraction of $\varphi(x)$ is arranged into the form

(2.9) $$\varphi(x)=\frac{x^2|}{|3}-\frac{1\cdot 3x^2|}{|5}-\frac{2\cdot 4x^2|}{|7}-\cdots-\frac{(n-1)(n+1)x^2|}{|2n+1}-\cdots$$

which coverges for all complex x except on the cuts along $(-\infty, -1)$ and $(1, \infty)$.[1)]

The nth approximative function $p_n(x)/q_n(x)$ of (2.9) is given by the following recurrence formula:

$$p_0=0,\ p_1=x^2,\ p_{n+1}=(2n+3)p_n-n(n+2)x^2p_{n-1}\ ,$$
$$q_0=1,\ q_1=3,\ q_{n+1}=(2n+3)q_n-n(n+2)x^2q_{n-1}\ .$$

For example, we have

$$\frac{p_2}{q_2}=\frac{5x^2}{3(5-x^2)},\ \frac{p_3}{q_3}=\frac{x^2(35-8x^2)}{15(7-3x^2)},\ \frac{p_4}{q_4}=\frac{7x^2(15-7x^2)}{15(21-14x^2+x^4)}\ .$$

(2.9) gives much better approximation than the original power series (2.6). In the interval $[-1, 1]$, the greatest error occurs at $x=\pm 1$, where the true value is $\varphi(\pm 1)=1/2$. The sum of the first n terms of (2.6) is

$$\frac{1}{2}\left(1-\frac{1}{2n+1}\right)$$

whose error is

(2.10) $$\frac{1}{2(2n+1)}\ .$$

On the other hand, it is not difficult to see by induction

$$p_n(\pm 1)=n!n(n+3)/4,\quad q_n(\pm 1)=(n+2)!/2\ ,$$

and then

$$\frac{p_n(\pm 1)}{q_n(\pm 1)}=\frac{n(n+3)}{2(n+1)(n+2)}=\frac{1}{2}-\frac{1}{(n+1)(n+2)}\ ,$$

whose error $1/(n+1)(n+2)$ is much smaller than (2.10).

Since $x=\pm 1$ are singularities of $\varphi(x)$, the error is fairly large there. Apart from the singularities, the approximation is better. For example at $x=1/2$, we have

$$p_3/q_3 \doteqdot 0.08800,\ p_4/q_4 \doteqdot 0.88020,$$

and the latter fits with the precise value of $\varphi(1/2)$ up to 6 digits.

1) Cf. e.g. [3] Chap. I.

3. Numerical approximation. In some problems, we need the integral value of $f(x)$ rather than the value of $f(x)$ itself. For such a purpose, the continued fraction is not suitable, especially if we integrate $f(x)$ through its singularity ± 1.

We need another numerical approximation formula.

The function $f(x)$ in (1.1) satisfies the functional relations

$$f(-x)=f(x),\quad f\left(\frac{1}{x}\right)+f(x)=1\ . \tag{3.1}$$

Hence, it will be better to take rational functions satisfying the same relations as (3.1). We put

$$g(x)=\sum_{n=1}^{N}\frac{c_n}{1+x^{2n}}\ , \tag{3.2}$$

where we must impose the condition

$$\sum_{n=1}^{N}c_n=1\ . \tag{3.3}$$

For the practical application, we choose $N=4$ considering the accuracy.

The function (3.2) has Maclaurin expansion

$$1-c_1x^2+(c_1-c_2)x^4+\cdots\ .$$

Comparing with (2.6), which is the expansion of $1-f(x)$, we first put

$$c_1=\frac{1}{3},\quad c_1-c_2=-\frac{1}{15},\quad \text{i.e.}\quad c_2=\frac{2}{5}\ . \tag{3.4}$$

Similar processes may be continued for c_3 and c_4, but we choose another determination for these coefficients.

Though the indefinite integral of $f(x)$ is not an elementary function, it is easy to see that

$$\int_0^1 f(x)dx=\frac{\pi^2}{16}+\frac{1}{4}\ ,\qquad \int_0^\infty f(x)dx=\frac{\pi^2}{8}\ . \tag{3.5}$$

Thus, we impose the condition

$$\int_0^\infty g(x)dx=\frac{\pi^2}{8}\ . \tag{3.6}$$

Since we have

$$\int_0^\infty \frac{dx}{1+x^{2n}}=\frac{\pi}{2n\ \sin(\pi/2n)}\ ,$$

the condition (3.6) reads

$$\frac{1}{2}c_1+\frac{1}{2\sqrt{2}}c_2+\frac{1}{3}c_3+\frac{\sqrt{2+\sqrt{2}}}{4\sqrt{2}}c_4=\frac{\pi}{8}\doteqdot 0.39270\,. \tag{3.7}$$

Inserting (3.4) and noting (3.3), we have linear equations

$$\begin{cases} c_3+0.97992c_4 & =0.25384 \\ c_3+c_4 & =0.26667(=4/15) \end{cases}$$

which gives the solution

$$c_3=-0.372\,,\qquad c_4=0.639\,.$$

We may take simply

$$c_3=-\frac{1}{3}\,,\qquad c_4=\frac{3}{5}\,.$$

Remember that the condition (3.3) must be strictly fulfilled, while the condition (3.6) may be satisfied approximately.

Starting the above values of c_n's, the author tried the improvement of the coefficients. Final results is as follows:

$$g(x)=\frac{0.33333}{1+x^2}+\frac{0.39344}{1+x^4}-\frac{0.34646}{1+x^6}+\frac{0.61969}{1+x^8}\,. \tag{3.8}$$

Though the function (3.8) is not a best-fit approximation, the error of the integral $\int_0^x (f(x)-g(x))dx$ is less than 10^{-4} for $x\geqq 3.5$, and less than 10^{-3} for $x\geqq 1.6$. (3.8) will be enough for practical applications.

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[3] A. N. Khovanskii (translated by P. Wynn), The application of continued fractions and their generalizations to problems in approximation theory, Nordhoff, 1963.